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GAMES OF INCOMPLETE INFORMATION, ERGODIC THEORY, AND THE MEASURABILITY OF EQUILIBRIA

BY

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ABSTRACT

We present an example of a one-stage three-player game of incomplete information played on a sequence space $\{0,1\}^{\mathbf{Z}}$ such that the players' locally finite beliefs are conditional probabilities of the canonical Bernoulli distribution on $\{0,1\}^{\mathbf{Z}}$, each player has only two moves, the payoff matrix is determined by the 0-coordinate and all three players know that part of the payoff matrix pertaining to their own payoffs. For this example there are many equilibria (assuming the axiom of choice) but none that involve measurable selections of behavior by the players. By measurable we mean with respect to the completion of the canonical probability measure, e.g., all subsets of outer measure zero are measurable. This example demonstrates that the existence of equilibria is also a philosophical issue.

1. Introduction

An equilibrium of a game is a set of strategies, one for each player, such that no player does better by choosing a different strategy, given that the other players do not change their strategies. In a game of incomplete information, what does

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it mean to do no better by choosing a different strategy? Should one evaluate a player's actions according to the subjective and local beliefs of that player, or should one evaluate according to a probability distribution determined objectively by the game?

When the subjective beliefs of the players are conditionals of a common prior, a central question is whether there is a difference between the equilibria defined according to the subjective local beliefs and the equilibria defined according to a global functional evaluation. The former we call **Bayesian** equilibria and the latter we call **Harsanyi** equilibria. Harsanyi [4] showed that these equilibria are equivalent when the set of all possible situations in the game is finite. Is there such an equivalence when the possible situations of the game are infinitely many?

In this paper we look at games that satisfy the following conditions:

(1) there is one stage of play with moves chosen simultaneously,

(2) there are finitely many players,

(3) each player has finitely many moves,

(4) there is a compact Polish space Ω with an atomless Borel probability distribution μ representing a choice by nature,

(5) for every combination of moves, one for each player, the payoff to each player is a continuous function from Ω to $\mathbf{R}$,

(6) at every point $x \in \Omega$ every player j has a discrete probability distribution on Ω, called his **belief**, with a finite support set $S^j(x)$ containing x such that at all the other points in this finite support set $S^j(x)$ the player j has the same discrete distribution,

(7) these discrete beliefs of the players change continuously (with respect to the weak* topology), and for any player j it forms a regular conditional probability of μ with respect to the sigma algebra $\mathcal{F}^j := \{B \mid B \text{ is Borel and } x \in B \Leftrightarrow S^j(x) \subseteq B\}$, and

(8) if B is a Borel set with $B \in \mathcal{F}^j$ for all players j then $\mu(B)$ is equal to either zero or one. (A **regular** conditional probability is a family of conditional probabilities, one for each Borel subset to be evaluated, however perceived also as a measurable function from the space Ω in question to the space of probability distributions on Ω.) Such games we call **ergodic** games.

A strategy for a player in an ergodic game is a function from Ω to the probability simplex of his moves that is constant within every finite support set that he could believe to be possible. We call a function on Ω **measurable** when it is measurable with respect to the completion of μ, meaning that all sets of outer measure zero are measurable.

We give an example of a three-player ergodic game that has no Bayesian equilibrium in measurable strategies (meaning also no Harsanyi equilibrium), yet it has many Bayesian equilibria that are not measurable. This example has the additional property that each player knows that part of the payoff matrix that pertains to his own payoff.

In the context of most multi-agent epistemic logics, a statement concerning the knowledge or belief of a player will correspond to a measurable subset of our sequence space. When applicable, the lack of a measurable equilibrium implies the impossibility for the behavior of the players in equilibrium to be determined by syntactic formulations of knowledge or belief.

The Cantor set is usually represented as a measure zero subset of the real numbers, obtained by removing central intervals of one-third length. However, one could also build the Cantor set as a subset with positive Lebesgue measure, by removing intervals of rapidly decreasing lengths. If, on every level of the construction, the size of every interval were the same then the canonical Bernoulli probability distribution on this Cantor set would be absolutely continuous with respect to the Lebesgue measure. Non-measurable subsets with respect to the Bernoulli distribution would be also non-measurable with respect to the Lebesgue measure. Because the existence of subsets of the real numbers that are not Lebesgue measurable can be denied by rejecting the axiom of choice [12], the existence of Bayesian equilibria for our game could be considered a philosophical question.

A zero-sum game is a two-person game such that for all possible situations and all combinations of moves, one for both players, the payoff for one player is the negation of the payoff for the other player. A zero-sum game is defined to have a value $r \in \mathbf{R}$ (as a payoff for the first player) when for every positive ϵ the first player has a strategy such that no matter what the second player does it guarantees to him an expected payoff of at least $r - \epsilon$, and vice versa, the second player has a strategy such that no matter what the first player does the payoff to the first player is held down to $r + \epsilon$ or less. Furthermore, if the zero-sum game has a value then a strategy of a player is **optimal** if it obtains for the player at least the value of the game no matter what his opponent does (meaning for the second player that the payoff to the first player is no more than the value of the game).

The problem with these definitions of value and optimality is how to define the expected payoff of an ergodic game. If the strategies are measurable, then a payoff can be defined as an expectation over the probability space. But if strategies are not measurable, meaning that the evaluations of the players are strictly local in character, how should we define the expected payoff?

If the players are restricted to measurable strategies then zero-sum ergodic games have values from optimal (measurable) strategies [6], though this will not be understandable from the exposition of this paper. The proof of the above follows Sion's Theorem [11], see also [9]. The general approach to establishing the existence of equilibria in non-zero-sum games is a fixed point argument involving compact strategy spaces and payoff functions continuous with respect to the product topology on these strategy spaces. To compactify the spaces of measurable strategies we can use the weak* topology; however, the weak* topology delivers only continuity with respect to changes in the behavior of one player. This partial continuity is sufficient for applying Sion's Theorem however insufficient in general for non-zero-sum games (as demonstrated by our example).

One approach to obtain positive results for non-zero-sum games has been to require from the game's information structure conditions implying that the payoff functions must be continuous with respect to the weak* topology. The first major result in this direction is by Milgrom and Weber [7]. One can perceive the probability space Ω as a subset of the product of spaces Ω_i, one for each player $i \in I$ and an additional Ω_0 to represent the choices of nature, such that Player i has identical beliefs at any pair of points in Ω if and only if they project to the same point of Ω_i. The probability distribution on Ω induces marginal probability distributions on the Ω_i. If the probability distribution on Ω is absolutely continuous with respect to the product measure induced by the marginals on Ω_i then Milgrom and Weber showed that there are equilibria in measurable strategies. While the absolutely continuous condition may appear at first to be natural, further reflection shows that it is highly restrictive. For example, any game for which there are two players with identical information violates this condition (which happens with our example).

The basic ideas in this paper belong to ergodic theory. One chooses a sequence space that allows for finitely many measure preserving involutions σ_i whose orbits are almost everywhere dense in the space. ($\sigma_i \circ \sigma_i$ is the identity, and for almost every x the subset of all the $\sigma_{i_1} \circ \sigma_{i_2} \circ \cdots \circ \sigma_{i_n}(x)$ is dense in the space.) To each player is associated a finite subset of involutions that commute with each other, and for every point x, this player believes that only the points in the finite orbit of his commuting involutions are possible. The entries in the payoff matrix are determined by the topological position in the sequence space, in particular, by some coordinate position. The conditions defining Bayesian equilibria pertain to the orbits of the involutions σ_i, and these orbits do not construct the space in a measurable way. However, the relationship between fixed point theory and ergodic theory in zero-sum ergodic games remains a mystery to us.

In the next section we define Bayesian and Harsanyi equilibria, prove that the countability of a belief structure implies the existence of Bayesian equilibria regardless of whether there exists a common prior, and we show that a Harsanyi equilibrium for an ergodic game generates a measurable Bayesian equilibrium for that game. In the third and last section we present the example of an ergodic game without measurable Bayesian equilibria.

2. Bayesian and Harsanyi equilibria

$\Delta(A)$ will stand for the set of regular Borel probability distributions of A, where A is a topological space (and if A is finite then we give A the discrete topology and $\Delta(A)$ is a simplex embedded in an Euclidean space). By the weak* topology on $\Delta(A)$ for a Polish space A we mean the weakest topology such that for every bounded continuous function $f\colon A \to \mathbf{R}$ the mapping from $\Delta(A)$ to $\mathbf{R}$ defined by the integration of f is also a continuous function. The origin of "weak*" comes from functional analysis: the same topology is called simply the weak topology by many working with probability theory.

Given a topological space Ω, a probability distribution $\mu \in \Delta(\Omega)$ and a Borel field $\mathcal{G}$, a function $f\colon \Omega \to \Delta(\Omega)$ is a regular conditional probability for μ and $\mathcal{G}$ when for every fixed Borel set A the function $f(\cdot)(A) \to [0,1]$ is measurable and for every pair $B \in \mathcal{G}$ and Borel A we have $\int_B f(z)(A)d\mu(z) = \mu(A \cap B)$ [2]. If Ω is a Polish space then always one can construct such a regular conditional probability ([3], Theorem 10.2.2).

The distance in an Euclidean space, including inside a simplex embedded in an Euclidean space, will be the Euclidean distance.

Throughout this paper we will assume the axiom of choice.

Sometimes a player will be referred to as *he* and sometimes as *she*.

2.1 MERTENS–ZAMIR SPACES. Ergodic games are a special case of games played on probability spaces that satisfy the Mertens–Zamir definition for a **belief space**.

A Mertens–Zamir belief space [5] is a tuple $(S, X, \psi, N, (t^j \mid j \in N))$, where X is a compact parameter set, S is a compact set, ψ is a continuous map from S to X, N is a finite set of players, for every $j \in N$ $t^j\colon S \to \Delta(S)$ is a continuous function (with respect to the weak* topology), and for every player j and every pair of points $s, s' \in S$ if $s' \in \text{support}(t^j(s))$ then $t^j(s) = t^j(s')$.

Define a **cell** of a Mertens–Zamir belief space to be a minimal set C with the property that at every point y in C every player's support set for the point y is contained in C (without the requirement that C must be compact).

Of special interest is the definition of mutual consistency for Mertens–Zamir belief spaces. For every player $j \in N$ define $\mathcal{T}^j$ to be the smallest Borel field of subsets of S such that the function t^j is measurable. A probability distribution μ on S is defined to be **consistent** if for every Borel subset $A \subseteq S$ we have that $\mu(A) = \int t^j(y)(A)d\mu(y)$. Mertens and Zamir [5] showed that consistency is equivalent to the stronger statement that for every $B \in \mathcal{T}^j$ and Borel subset $A \subseteq S$ we have $\mu(A \cap B) = \int_B t^j(y)(A)d\mu(y)$. We prefer to use the Borel field $\mathcal{F}^j$ defined by $\mathcal{F}^j := \{B | \ B \text{ is Borel and } x \in B \Leftrightarrow \text{support}(t^j(x)) \subseteq B\}$, the largest Borel field on which a Borel measurable strategy for Player j can be defined.

For Mertens–Zamir belief spaces it is easy to construct games that are played on the space S. Let ψ: $S \to X$ be the continuous function, N the finite player set, and for each $j \in N$ t^j: $S \to \Delta(S)$ the subjective beliefs of Player j. For each player j, there is a finite action set A^j with $n^j := |A^j|$. X corresponds to a compact subset of $n^1 \times \cdots \times n^{|N|}$ matrices $(Q_x | \ x \in X)$; every entry of every matrix is a vector payoff for the players in $\mathbf{R}^N$. Nature chooses a point in S according to the common prior μ, which means also that a parameter in X is chosen through the function ψ. The players choose moves in their respective A^j independently, and after the choices are made the payoff to the players is the vector entry in Q_x corresponding to nature's choice of the parameter in X and the moves of the players. A Bayesian equilibrium for a point z in the belief space is an $|N|$-set of functions $(f^j | \ j \in N)$, each f^j from the cell that contains z to $\Delta(A^j)$, the simplex of mixed strategies, with the following properties for every player $j \in N$

(1) f^j is constant within all support sets of Player j,

(2) for all $j' \neq j$ within the support set of $t^{j'}(z)$ the function f^j is measurable with respect to the completion of the probability distribution $t^{j'}(z)$, and

(3) within the support set of $t^j(z)$ Player j can do no better than $f^j(z) \in \Delta(A^j)$ in response to the other functions $f^{j'}, j' \neq j$, as evaluated by $t^j(z)$.

When the $|N|$-set of functions is a Bayesian equilibrium for all points in a cell, then we call it a **cellular** equilibrium. A Bayesian equilibrium for the whole space is a collection of **cellular** equilibria, once for each cell.

If, additionally, the Mertens–Zamir belief space has a consistent common prior μ, we define a Harsanyi equilibrium to be a set of functions

$$(f^j \colon S \to \Delta(A^j)) | \ j \in N),$$

each f^j **measurable** with respect to the Borel field $\mathcal{F}^j$, such that no player can attain a higher expected payoff as evaluated by μ by choosing another such measurable function (given that the strategies of the other players do not change).

Notice that ergodic games are special cases of games played on Mertens–Zamir spaces with consistent common priors. From now on we will use the terminology of Mertens–Zamir spaces to describe an ergodic game, in particular the t^j to represent the belief of Player j; we will write support $(t^j(x))$ instead of $S^j(x)$. We will adopt the term **cell**, though for our example a cell will be an orbit of a non-abelian group acting on the probability space. (A confusion with the orbits of a shift transformation T is avoided.) In an ergodic game, the local measurability condition (2) of a Bayesian equilibrium disappears, leaving Bayesian equilibria with no measurability conditions at all.

2.2 Locally finite games. We define **locally finite** games. Unlike ergodic games, there will be no assumption of a common prior or a compact space.

Let S be a finite or countably infinite set. There is a finite or countably infinite collection $\mathcal{J}$ and a map $\nu\colon \mathcal{J} \to 2^S$ such that for every $J \in \mathcal{J}$, $\nu(J)$ is a subset of S, and for every $s \in S$ the point s is contained in $\nu(J)$ for only finitely many members J of $\mathcal{J}$. A member of $\mathcal{J}$ is a player of our game. For every $J \in \mathcal{J}$ there is a finite set A^J of moves and a sigma-additive probability distribution p^J on the set $\nu(J)$ such that the support of p^J is the set $\nu(J)$. For every $s \in S$ define $\mathcal{J}(s)$ to be the finite subset of $\mathcal{J}$ whose projections by ν contain the point s. For every $s \in S$ there is a payoff matrix Q_s of size $\times_{J\in\mathcal{J}(s)}|A^J|$ with entries in $\mathbf{R}^{\mathcal{J}(s)}$; for every choice of $a \in \prod_{J\in\mathcal{J}(s)} A^J$ there are corresponding payoffs for all the players in $\mathcal{J}(s)$. For every $s \in S$ and every player $J \in \mathcal{J}(s)$ let $M_s^J \geq 1$ be an upper bound on the absolute values of all payoffs to J in the matrix Q_s, and we assume that for every player $J \in \mathcal{J}$ the sum $\sum_{s\in\nu(J)} p^J(\{s\})M_s^J$ is finite.

A strategy of a player $J \in \mathcal{J}$ is a member of $\Delta(A^J)$. A Bayesian equilibrium for a locally finite game is a set of strategies $(f^J \in \Delta(A^J)|\ J \in \mathcal{J})$, such that for every $J \in \mathcal{J}$ all moves in the support of f^J deliver the same expected payoff for the player J and, furthermore, he cannot get a higher expected payoff by choosing a move in A^J not in the support of f^J (given that the other players remain with $(f^K \in \Delta(A^K)|\ K \neq J,\ K \in \mathcal{J})$), with the expected payoff of a move calculated according to the distribution p^J and the expected payoffs from the matrices Q_s at each of the states $s \in \nu(J)$.

Every cell of an ergodic game will define a locally finite game in a natural way. Every distinct support set of a player in the cell (of an ergodic game) can be viewed as a distinct player of the corresponding locally finite game. The cell of an ergodic game is countable in size because it is the countable union of finite support sets. The continuity of the payoff functions defined on the compact set Ω establishes a uniform bound for the payoffs.

PROPOSITION 1: *For every locally finite game there exists a Bayesian equilibrium.*

Proof: Let the set S be enumerated by $S = \{s_1, s_2, s_3, \ldots\}$. For every $i = 1, 2, \ldots$ define the game Γ_i to be that played on the set $S_i := \{s_1, \ldots, s_i\}$ with the players $\mathcal{J}_i := \{J \in \mathcal{J} \mid \nu(J) \cap S_i \neq \emptyset\}$ and the probability distributions p_i^J on S_i (for the players in $\mathcal{J}_i$) induced by $p_i^J(B) := p^J(B)/p^J(S_i)$ for all $B \subseteq S_i$. Since each S_i and $\mathcal{J}_i$ are finite, there exists a Nash equilibrium $\tilde{f}_i := (f_i^J \in \Delta(A^J) \mid J \in \mathcal{J}_i)$ to the game Γ_i for every $i \geq 1$ [8]. Assigning any distribution to f_i^J when $\nu(J)$ has an empty intersection with S_i, we have a sequence f_i in the set

$$\tilde{A} := \prod_{J \in \mathcal{J}} \Delta(A^J).$$

We give $\tilde{A}$ the product topology. Due to Tychanov's Theorem, $\tilde{A}$ is compact, and we can assume that there exists a convergent subsequence f_{i_n} of the f_i converging to $f \in \tilde{A}$. Redefine the sequence $(f_i \mid i = 1, 2, \ldots)$ so that the new f_n is the old f_{i_n}.

We aim to show that f defines a Bayesian equilibrium of the original game. Fix an $\epsilon > 0$. We will show that f is an ϵ-Bayesian equilibrium.

For every $i \geq 1$ let N_i be the finite cardinality of the set $\mathcal{J}_i$, let B_i be the maximal size of A^K for any Player K in $\mathcal{J}_i$, and for any Player J in $\mathcal{J}_i$ let M_i^J be the maximal size of the $M_{s_1}^J, M_{s_2}^J, \ldots, M_{s_i}^J$.

Let Player $J \in \mathcal{J}$ be given. Let i_0 be so large that $\sum_{s \in S_{i_0}} p^J(\{s\}) M_s^J > -\epsilon/5 + \sum_{s \in \nu(J)} p^J(\{s\}) M_s^J$. Let δ be the smallest positive probability by which Player J chooses some move with the strategy f^J. Now choose an $i_1 > i_0$ so large that $l \geq i_1$ implies that $f_l^K \in \Delta(A^k)$ is within $\min(\epsilon/(4M_{i_0}^J B_{i_0} N_{i_0}), \delta/2)$ of f^K for all the $K \in \mathcal{J}_i$, including J. Given that all other players $K \neq J$ stay with their strategies f^K, we need only show that all moves in support (f^J) give to Player J within ϵ of the maximum payoff obtainable from any of his moves. Due to the definition of δ and the condition involving δ that defines i_1 we have that support (f^J) is contained in support (f_l^J) for any $l \geq i_1$. Because of the other condition defining i_1, namely that for all $l \geq i_1$ and any other player K the strategy f_l^K is within $\epsilon/(4M_{i_0}^J B_{i_0} N_{i_0})$ of the limit strategies f^K, for any fixed move of Player J the payoff consequence to Player J conditioned on membership in the set S_{i_0} if the other players choose $f_{i_1}^K$ does not differ by more than $\epsilon/4$ from the same payoff consequence to Player J if the other players chose instead f^K. Therefore, the equilibrium property of the $f_{i_1}^K$ (including $K = J$) and the fact that the total payoff for Player J from states outside of S_{i_0} does not exceed $\epsilon/5$ suffice for our conclusion. ∎

2.3 Measurable equilibria. The following proof was explained to me by J.-F. Mertens.

Proposition 2: *Any Harsanyi equilibrium of an ergodic game will generate a measurable Bayesian equilibrium of that game.*

Proof: For all $j \in N$ assume that $f^j \colon \Omega \to \Delta(A^j)$ is a Harsanyi equilibrium, meaning also that they are Borel measurable functions (measurable with respect to the Borel fields $\mathcal{F}^j$, respectively). Let μ be the common prior probability distribution on Ω. For all $j \in N$, all moves $a \in A^j$, and all positive integers m define $W_0^j(a,m)$ to be the subset of Ω such that Player j can obtain a payoff of at least $1/m$ more by choosing the move a instead of the distribution determined by f^j, as evaluated by the subjective probability distribution t^j. Since all the f^k are Borel measurable and t^j is continuous in the weak* topology, the sets $W_0^j(a,m)$ are also Borel. (Due to the continuity of t^j, it suffices to show for every fixed and bounded Borel measurable function $f \colon \Omega \to \mathbf{R}$ that

$$\{\nu \in \Delta(\Omega) \mid \int f d\nu \geq 1/m\}$$

is a Borel subset of $\Delta(\Omega)$. This follows by Luzin's Theorem, since we can approximate f in measure by a sequence of continuous functions.) Since the payoff evaluation of Player j is constant on all support sets of t^j, the sets $W_0^j(a,m)$ are also in $\mathcal{F}^j$. If $\mu(W_0^j(a,m))$ were positive for some j, a, and m, then the $(f^j \mid j \in N)$ would not have been a Harsanyi equilibrium. We define W_0 to be $\bigcup_{j,a,m} W_0^j(a,m)$. Because this union is countable, the set W_0 is also Borel of measure zero. Now for all $l \geq 1$ define inductively the sets

$$W_l^j := \{x \in \Omega \mid t^j(x)(W_{l-1}) > 0\} \quad \text{and} \quad W_l := \bigcup_{j \in N} W_l^j.$$

We claim for all l and j that W_l^j is in $\mathcal{F}^j$, W_l is Borel, and $\mu(W_l) = 0$. We proceed by induction, assuming the claim for $l-1$. That W_{l-1} is Borel implies that W_l^j is in $\mathcal{F}^j$, and this implies that $W_l = \bigcup_{j \in N} W_l^j$ is Borel. Due to the formula $\mu(W_{l-1} \cap W_l^j) = \int_{W_l^j} t^j(y)(W_{l-1}) d\mu(y)$ and the fact that $t^j(y)(W_{l-1}) > 0$ for all $y \in W_l^j$, $\mu(W_l^j) > 0$ would imply that $\mu(W_{l-1}) \geq \mu(W_{l-1} \cap W_l^j) > 0$, a contradiction. Therefore we can assume also that $\mu(W_l) = 0$.

Define $W := \bigcup_{l=0}^{\infty} W_l$. We have two important properties, that W is Borel with $\mu(W) = 0$, and also from the structure of ergodic games that W is the union of cells. We can alter our Harsanyi equilibrium. We keep the original functions on $\Omega \backslash W$, and for all the cells in the set W we introduce any Bayesian equilibria obtained from Proposition 1. The result is a measurable Bayesian equilibrium that we seek.

3. Many Bayesian equilibria, none measurable

Before presenting our example, we need the following lemma.

LEMMA 0: *Let Ω be a Polish space and let ϕ be a continuous measure preserving involution with respect to $\mu \in \Delta(\Omega)$. Define the Borel field*

$$\mathcal{G} := \{B|\ B \subseteq \Omega \text{ is Borel and } x \in B \Leftrightarrow \phi(x) \subseteq B\}.$$

Define a function $t : \Omega \to \Delta(\Omega)$ by $t(x)(B) =$

0 if both x and $\phi(x)$ are not in B,

1 if both x and $\phi(x)$ are in B, and

1/2 otherwise.

t is continuous with respect to the weak* *topology and the function t: $\Omega \to \Delta(\Omega)$ is a regular conditional probability induced by μ and $\mathcal{G}$.*

Proof: That t changes continuously in the weak* topology follows directly from the continuity of ϕ.

Next we must show for a fixed Borel set A and value $r \in [0,1]$ that the set $\{z|\ t(z)(A) > r\}$ is in $\mathcal{G}$. We consider two cases: $r < 1/2$ and $r \geq 1/2$. If $r < 1/2$ then the above set is $A \cup \phi(A)$, which is in $\mathcal{G}$ because ϕ is a continuous involution. If $r \geq 1/2$ then the above set is $A \cap \phi(A)$, which is in $\mathcal{G}$ for the same reasons.

Second, we will show for every $B \in \mathcal{G}$ and Borel set A that $\mu(B \cap A) = \int_B t(z)(A)d\mu(z)$. Let B be any member of $\mathcal{G}$ and z any member of B.

Let B_0 be the subset of the z where $\phi(z) = z$, B_1 the subset where either z or $\phi(z)$ is in A but not both, B_2 the subset where $z \neq \phi(z)$ and both z and $\phi(z)$ are in A, and B_3 the subset where $z \neq \phi(z)$ and neither z nor $\phi(z)$ is in A.

Since all sets are Borel, we can write $\int_B t(z)(A)d\mu(z)$ as $\int_{B_0} t(z)(A)d\mu(z) + \int_{B_1} t(z)(A)d\mu(z) + \int_{B_2} t(z)(A)d\mu(z) + \int_{B_3} t(z)(A)d\mu(z)$.

CASE 0: Since $t(z)$ is the function 1_A in B_0, $\int_{B_0} t(z)(A)d\mu(z) = \mu(A \cap B_0)$.

CASE 1: Notice that $z \in B_1$ if and only if $\phi(z) \in B_1$. Since $t(z)(A) + t(\phi(z))(A) = 1/2 + 1/2 = 1$ for all $z \in B_1$, $\int_{B_1}(t(z)(A) + t(\phi(z))(A))d\mu(z)$ is equal to $\int_{B_1} d\mu(z) = \mu(B_1)$, but also to $2\int_{B_1} t(z)(A)d\mu(z)$ from the measure preserving property of ϕ and the fact that $t(z)(A)$ is a constant $1/2$ for all $z \in B_1$. Again from the measure preserving property we have that $\mu(A \cap B_1) = \mu(\phi(A \cap B_1))$. But $\phi(A \cap B_1)$ is exactly $B_1 \backslash A$, and therefore $2\mu(A \cap B_1) = \mu(B_1)$ and $\int_{B_1} t(z)(A)d\mu(z) = \mu(A \cap B_1)$, as desired.

CASE 2: $t(z)(A) = t(\phi(z))(A) = 1$ for all $z \in B_2$ and therefore $\int_{B_2} t(z)(A)d\mu(z) = \int_{B_2} d\mu(z) = \mu(B_2) = \mu(A \cap B_2)$.

CASE 3: If neither $z \in A$ nor $\phi(z) \in A$ then $t(z)(A) = t(\phi(z))(A) = 0$ and therefore $0 = \int_{B_3} t(z)(A)d\mu(z) = \mu(A \cap B_3)$. ∎

3.1 THE EXAMPLE. Let a and b be two distinct states of nature. Define Ω to be $\{a, b\}^{\mathbf{Z}}$, where $\mathbf{Z}$ is the set of integers, including both the positive and the negative. The 0-coordinate of a point in Ω determines the state of nature, so that if $y \in \Omega$ and $y^0 = a$ then a is the state of nature at the point y. We define μ to be the canonical Bernoulli distribution on Ω, giving equal probability independently to a and b in all coordinate positions.

There will be three players, Players One, Two, and Three. Let $\sigma\colon \Omega \to \Omega$ be the measure preserving involution defined by $(\sigma(y))^i := y^{-i}$, where x^i is the ith coordinate of $x \in \Omega$; σ is the reflection of the doubly infinite sequence about the position zero. Let $\tau\colon \Omega \to \Omega$ be the measure preserving involution defined by $(\tau(y))^i := y^{1-i}$. It follows that $T := \tau \circ \sigma$ is the usual Bernoulli shift operator $(T(y))^i = y^{i-1}$. Define t^1 and t^2 according to Lemma 0; this means that at any point $y \in \Omega$ Player One considers only y and $\sigma(y)$ to be possible, and with equal probability (if $\sigma(x) = x$ then Player One believes in x with full probability). At any x, both Player Two and Player Three believe that only x and $\tau(x)$ are possible, and with equal probability. The cells will be the orbits of the involutions σ and τ. Players Two and Three have the same beliefs. Player One always knows the state of nature, a or b, but not always what the other players might know. Such a game is called a game of incomplete information *on one and a half sides* [13], though this term was invented with reference to infinitely repeated zero-sum games.

Now we define the moves and payoffs. Each player has two moves, L and R, standing for left and right. The game is a variation of the well known game of matching pennies, with Player One playing against both Player Two and Player

Three. Players Two and Three want to match the pennies, Player One wants to have a mismatch. The differences to the conventional game of matching pennies have two aspects. First, Player Two has a special relationship to the move L, her favorite move, and likewise Player Three has a special relationship to her favorite move R. Second, if both Players Two and Three choose their favorite moves, then the payoff to Player One is dependent on the state of nature.

By playing L, Player One at $x \in \Omega$ receives

-1 if Player Two and Player Three both choose L,
1 if Players Two and Three both choose R,
0 if Player Two chooses R and Player Three chooses L, and
$\delta_a(x^0)$ if Player Two chooses L and Player Three chooses R.

δ stands for the Kroniker delta, which means that $\delta_a = 1$ if the state of nature is a and $\delta_a = 0$ if the state of nature is b.

By playing R, Player One at $x \in \Omega$ receives

1 if Player Two and Player Three both choose L,
-1 if Player Two and Three both choose R,
0 if Player Two chooses R and Player Three chooses L, and
$\delta_b(x^0)$ if Player Two chooses L and Player Three chooses R.

By playing L, Player Two receives (independently of the state of nature)

2 if Player One and Player Three choose L,
1 if Player One chooses L and Player Three chooses R,
-1 if Player One chooses R.

By playing R, Player Two receives

-1 if Player One chooses L, and
1 if Player One chooses R.

By playing L, Player Three receives

1 if Player One chooses L, and
-1 if Player One chooses R.

By playing R, Player Three receives

-1 if Player One chooses L,

1 if Player One chooses R and Player Two chooses L, and

2 if Player One and Player Two choose R.

For all Players $i = 1, 2, 3$ let $f^i \colon \Omega \to [0, 1]$ be the behavior strategy for Player i, with $f^i(x)$ representing the probability at $x \in \Omega$ that Player i chooses the move L. The only a-priori requirement placed on the behavior strategies $(f^i |\ i = 1, 2, 3)$ is that for all $x \in \Omega$, $f^1(x) = f^1(\sigma(x))$ and $f^j(x) = f^j(\tau(x))$ for either $j = 2, 3$.

Given any Player $j \in \{1, 2, 3\}$ and two functions $(f^i |\ i \neq j)$ for the other two players, one can calculate the expected payoff to Player j at every $x \in \Omega$ for choosing L and for choosing R; for Player One one must average the expected payoffs of the moves with respect to $(f^i(x) |\ i \neq 1)$ and $(f^i(\sigma(x)) |\ i \neq 1)$, and for Players Two and Three one must do the same with respect to $(f^i(x) |\ i \neq j)$ and $(f^i(\tau(x)) |\ i \neq j)$ for $j = 2, 3$. The three functions $(f^i |\ i = 1, 2, 3)$ are a Bayesian equilibrium for the game if and only if at every appropriate pair of points for every j if Player j chooses the move $D \in \{L, R\}$ with positive probability according to f^j then the other move $E \in \{L, R\}$, $E \neq D$ must deliver to Player j no higher an expected payoff than does the move D.

Define a pair of strategies $(f^2, f^3) \in [0, 1]^2$ for Players Two and Three to be a **balanced** pair if and only if

$$f^2 = 1 \text{ and } f^3 < 1 \quad \text{or}$$
$$f^3 = 0 \text{ and } f^2 > 0.$$

The balanced pairs are those for which at least one of the two players chooses her favorite move with certainty and the other player is not choosing the same move with certainty.

The two pairs of strategies $(f^2, f^3) \in [0, 1]^2$ such that $f^2 = f^3 = 0$ or $f^2 = f^3 = 1$ will be called the **coordinated** pairs. A strategy $f^1 \in [0, 1]$ of Player One that takes on the value of 0 or 1 will be called a **pure** strategy. A strategy of Player One in $[0, 1]$ that is not pure will be called **mixed**.

LEMMA 1: *In any Bayesian equilibrium behavior Players Two and Three are using only pairs that are balanced or coordinated.*

Proof: Suppose for the sake of contradiction that $f^2(x) < 1$ and $f^3(x) > 0$. If $f^1(x) + f^1(\tau(x)) \geq 1$ then Player Two would prefer to choose L ($f^2 = 1$) at the pair x and $\tau(x)$, and if $f^1(x) + f^1(\tau(x)) \leq 1$ then Player Three would prefer to choose R ($f^3 = 0$) at the pair x and $\tau(x)$. ∎

Although the strategies available a-priori to Players Two and Three are two-dimensional, in Bayesian equilibrium only a one-dimensional subset will be used. This will allow us to perceive the game much like a two-by-two game played between two players. Notice that if both Players Two and Three choose their favorite moves with certainty in a Bayesian equilibrium then it is impossible for either player to prefer her favorite move over the other move (since otherwise the other player would also prefer her non-favorite move). For this reason we perceive the pair $(f^2 = 1, f^3 = 0)$ as balanced, although strictly speaking both players are choosing pure strategies.

3.2 STRATEGY OF THE PROOF. We suppose for the sake of contradiction that the three functions $f^i \colon \Omega \to [0,1]$ for $i = 1,2,3$ are measurable behavior strategies in Bayesian equilibrium. We will show that this leads to a contradiction. From Proposition 1 there would exist non-measurable Bayesian equilibria.

Our strategy is the following. We will divide Ω into two parts. The first part will be the set of $x \in \Omega$ where Player One is using a mixed strategy and Players Two and Three are using a balanced strategy; the second part will be its complement. The measurability assumption will imply that both parts of Ω are of measure zero.

From ergodic theory, we will need a few well known results.

First, we need the Birkhoff Ergodic Theorem. If a transformation T on a probability space Ω is measure preserving and f is an integrable function on Ω then the Birkhoff Ergodic Theorem states that $\lim_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n-1} f(T^i(x))$ converges almost everywhere to an integrable function f^* such that the integral of f^* over Ω is equal to that of f ([14], Theorem 1.14).

Second, we need the property called **ergodic**. A measure preserving transformation T of a probability space with a probability measure μ is ergodic if the only measurable sets B with the property $T^{-1}(B) = B$ satisfy $\mu(B) = 0$ or $\mu(B) = 1$. A measure preserving transformation is ergodic if and only if the only measurable sets B with $\mu(T^{-1}(B)\Delta B) = 0$ are those with $\mu(B) = 0$ or $\mu(B) = 1$ ([14], Theorem 1.5), where here Δ stands for the symmetric difference. A measure preserving transformation T is **mixing** if for every pair A and B of measurable sets $\lim_{n\to\infty} \mu(A \cap T^{-n}(B)) = \mu(A)\mu(B)$. The Bernoulli shift T of our probability space is mixing and mixing is a stronger property than ergodicity ([1], pages 11–12), therefore T^n is both mixing and ergodic for all non-zero integers n. We will use that T^2 is ergodic. The ergodicity of T demonstrates that our example satisfies Condition (8) for being an ergodic game — any Borel set B such that $x \in B \Leftrightarrow \sigma(x) \in B$ and $x \in B \Leftrightarrow \tau(x) \in B$ will also satisfy $x \in B \Leftrightarrow T(x) \in B$.

The cells partition the space Ω. Because the σ and τ are measure preserving, the smallest union of cells containing any measure zero subset of Ω is also of measure zero. Notice that for almost all points $x \in \Omega$ the points $T^k(x)$ and $T^k \circ \tau(x)$ for all $k \in \mathbf{Z}$ are distinct and comprise the cell containing x. Define such points and their cells to be **doubly infinite**.

For any finite $k \geq 1$ any measurably defined behavior that occurs at most k times in any doubly infinite cell must occur only in a set of measure zero, since otherwise the distinctness of the points $T^k(x)$ and $T^k \circ \tau(x)$ for almost all $x \in \Omega$ would imply that the space Ω has infinite measure.

Define a point $x \in \Omega$ to be **normal** if and only if Player One uses a mixed strategy at x and Players Two and Three use a balanced strategy at x. Define a cell to be normal if and only if it contains at least one normal point. The other cells will be called **abnormal**.

3.3 Normal cells. We define a homeomorphism between the balanced pairs and the real numbers. Define k: $(0,1] \to \mathbf{N_0} = \{0,1,2,\ldots\}$ and s: $(0,1] \to [0,1]$ by $r = 3^{-k(r)}(1 - 2s(r)/3)$ for $k(r)$ being the last number such that $3^{-k(r)} \geq r$. Define W to be the set of balanced pairs, $W := \{(f^2, f^3) |\ f^2 = 1 \text{ or } f^3 = 0\} \backslash \{(0,0),(1,1)\}$. Define the homeomorphism ϕ: $W \to \mathbf{R}$ by

$$\begin{aligned} \phi(1,0) &:= 0, \\ \phi(1,t) &:= k(1-t) + s(1-t), \quad \text{and} \\ \phi(t,0) &:= -s(t) - k(t). \end{aligned}$$

Lemma 2: *If x is doubly infinite, $f^1(x)$ is mixed and the strategy pair $(f^2(x), f^3(x))$ is balanced, then there is one and only one value for $f^1(\tau x)$ that preserves the Bayesian equilibrium property for Players Two and Three and there is one and only one balanced pair $(f^2(\sigma(x)), f^3(\sigma(x)))$ that preserves the Bayesian equilibrium property for Player One, namely $f^1(\tau(x)) = 1 - f^1(x)$ and*

$$(f^2(\sigma(x)), f^3(\sigma(x))) = \phi^{-1}(-1^{\delta_b(x^0)} - \phi(f^2(x), f^3(x))).$$

Proof: By symmetry we assume that $f^2(x) = 1$. The condition that $f^1(\tau(x)) = 1 - f^1(x)$ follows by the indifference of Player Three (and for both players in the case that $f^3(x) = 0$, as discussed above).

Again, assuming $f^2(x) = 1$, we must divide the argument into three cases. Case A is that of $0 \leq f^3(x) \leq 2/3$ and $x^0 = a$, Case B is that of $x^0 = b$, and Case

C is that of $2/3 \leq f^3(x) < 1$ and $x^0 = a$. By Lemma 1 we need only consider the balanced pairs and the coordinated pairs. Notice that increasing values for ϕ in both cases of $x^0 = a$ and $x^0 = b$ imply an increasing preference by Player One for the move R, so that if we find one balanced pair that delivers indifference to Player One then we have found the only such balanced pair, and furthermore neither coordinated pair could deliver such an indifference.

CASE A: Consider the strategy pair $g^2 = 1$ and $g^3 = 2/3 - f^3(x)$, which satisfy $\phi(f^2(x), f^3(x)) + \phi(g^2, g^3) = 1$. If Player One chooses L, he can expect a payoff of $\frac{1}{2}(-f^3(x) + 1 - f^3(x) - g^3 + 1 - g^3) = 1/3$. By playing R he can expect a payoff of $\frac{1}{2}(f^3(x) + g^3) = 1/3$.

CASE B: Consider the strategy pair $g^3 = 0$ and $g^2 = (1 - f^3(x))/3$, which satisfy $\phi(f^2(x), f^3(x)) + \phi(g^2, g^3) = -1$. If Player One chooses L, he can expect a payoff of $\frac{1}{2}(-f^3(x) + 1 - g^2) = (1 - f^3(x))/3$. If Player One chooses R, he can expect a payoff of $\frac{1}{2}(f^3(x) + 1 - f^3(x) - (1 - g^2) + g^2) = (1 - f^3(x))/3$.

CASE C: Consider the strategy pair $g^3 = 0$ and $g^2 = 3(1 - f^3(x))$, which satisfy $\phi(f^2(x), f^3(x)) + \phi(g^2, g^3) = 1$. If Player One chooses L, he can expect a payoff of $\frac{1}{2}(-f^3(x) + 1 - f^3(x) + g^2 + 1 - g^2) = 1 - f^3(x)$. By playing R he can expect a payoff of $\frac{1}{2}(f^3(x) - (1 - g^2)) = 1 - f^3(x)$. ∎

LEMMA 3: *The measure of the union of all normal cells must be zero or one.*

Proof: By Lemma 2 a doubly infinite cell is normal (contains a normal point) if and only if all the points in the cell are normal. This means that the set of normal points is a T-invarient set, and therefore by the ergodicity of T they must be of measure zero or one. ∎

LEMMA 4: *Let $f\colon \Omega \to \mathbf{R}$ be a function with $f(Tx) + f(x) = -1^{\delta_b(x^0)}$ for almost all $x \in \Omega$.* **Conclusion:** *The function f cannot be measurable.*

Proof: Suppose for the sake of contradiction that the function f is measurable. This would imply that there exists an $M > 0$ such that the probability that f is in $[-M + 2, M - 2]$ is at least $9/10$.

Now consider the function $g_M\colon \Omega \to [0, 1]$ defined by

$$g_M(x) := \lim_{n\to\infty} \frac{1}{n+1} |\{k \mid 0 \leq k \leq n, |f(T^k x)| \geq M\}|.$$

Now consider the transformation T^2. The action of T^2 is that of a random walk, with

$$f(T^2x) - f(x) = 2 \text{ if } x^0 = b \text{ and } x^{-1} = a,$$
$$f(T^2x) - f(x) = -2 \text{ if } x^0 = a \text{ and } x^{-1} = b, \text{ and}$$
$$f(T^2x) = f(x) \text{ if both } x^0 \text{ and } x^{-1} \text{ have the same value.}$$

Therefore, for almost all x there is a $k \in \mathbf{Z}$ with $f(T^kx) > M + 1$, which means that the expected value of g_M is at least 1/2. But by the Birkhoff Ergodic Theorem, the expected value of g_M must equal the probability in Ω that the function f exceeds the value of M, a contradiction. ∎

PROPOSITION 3: *The union of all normal cells is of measure zero.*

Proof: By Lemma 2 the function $\phi(f^2(x), f^3(x))$ obeys the property of Lemma 4. Since we must assume that this function is measurable, by Lemma 4 it cannot be defined on a subset of measure one. By Lemma 3 this implies that the union of normal cells has zero measure. ∎

3.4 ABNORMAL CELLS. We proceed with the assumption, justified by Proposition 3, that the union of all abnormal cells has measure one.

Define $x \in \Omega$ to be a **pure** point if and only if $f^i(x) \in \{0, 1\}$ for all $i = 1, 2, 3$ with $f^2(x) = f^3(x)$ (meaning that Player One uses a pure strategy at x and Players Two and Three use a coordinated pair). Define the **wind** direction from a doubly infinite pure point x to be the direction on the cell of the y adjacent to x (equal to either $\sigma(x)$ or $\tau(x)$) such that $y = \sigma(x)$ if all three of the $f^i(x)$ have the same value in $\{0, 1\}$, and otherwise $y = \tau(x)$ if Player One chooses a different move from that chosen by Players Two and Three. Define the **lee** direction from a doubly infinite pure point to be the direction opposite to the wind direction. Define a pure point x to be a **book end** if additionally either Player One at x has a clear preference for his chosen move or the player among Players Two or Three who is not choosing her favorite move has a clear preference for her chosen and non-favorite move.

LEMMA 5:

(1) *The behavior of all players on the wind side of a doubly infinite pure point is determined; all of the points on the wind side of a doubly infinite pure point are also pure, alternating in values, and none are book ends.*

(2) *There can be at most two book ends in any doubly infinite abnormal cell.*

(3) *If there are no book ends in a doubly infinite abnormal cell, then either Player One or Players Two and Three are performing alternating behavior, meaning that choosing any point x in the cell we have that either the value $f^1(T^n(x)) \in \{0,1\}$ is determined by the parity of n or the value $f^2(T^n(x)) = f^3(T^n(x)) \in \{0,1\}$ is determined by the parity of n.*

Proof: (1) By symmetry, we can assume that Players Two and Three choose the move L at x. We consider two cases, (A) that Player One chooses L and (B) that Player One chooses R. In Case (A) the point $\sigma(x)$ (and the transformation T) defines the wind direction. If Players Two and Three did not both choose R at $\sigma(x)$, then it would have been in the interest of Player One to choose R at x. In Case (B) the point $\tau(x)$ (and T^{-1}) defines the wind direction. If Player One did not choose L at the point $\tau(x)$ then it would have been in the interest of Player Three to choose R at x. The rest follows by induction.

(2) For the sake of contradiction, let us assume that there are at least three book ends x in a doubly infinite cell. One of these three book ends must be between the other two. By Part 1, in one of the two directions from this middle book end there are no book ends, a contradiction.

(3) We consider three cases, (A) that there are no pure points in the cell and that Player One chooses a pure strategy at x, (B) that there are no pure points in the cell and Players Two and Three choose a coordinated pair at x, and (C) x is a pure point in the cell.

Case (A): If Player One is choosing L at x, then the lack of a pure point implies that Players Two and Three are choosing balanced strategies at both x and $\sigma(x)$. To maintain the indifference of these balanced choices by at least one of either Players Two or Three, Player One must choose R at Tx and $T^{-1}x$. The rest follows by induction.

Case (B): If Players Two and Three are both choosing L at x, then the lack of a pure point implies that Player One is choosing mixed strategies at both x and $\tau(x)$. To maintain the indifference of Player One, Players Two and Three are both choosing R at both Tx and $T^{-1}x$. The rest follows by induction.

Case (C): Assume that x is a pure point, that Players Two and Three are choosing L at x, and Player One is choosing either L or R at x. If Player One is choosing L at x, then the lack of a book end implies that Player One is choosing R at $T^{-1}(x)$, in the lee direction of x. If Player One is choicing R at x, then the lack of a book end implies that Players Two and Three are both choosing R at

$T(x)$, in the lee direction of x. By induction and Part (1) we conclude that the whole cell consists of alternating behavior by all three players. ∎

PROPOSITION 4: *The union of all the abnormal cells is a measure zero set.*

Proof: Due to Part (2) of Lemma 5, the measure of the set of all doubly infinite book ends is zero, and therefore also of all the cells containing a book end. We proceed with the assumption (also supported by Proposition 3) that almost all of Ω is contained in cells without any book ends.

Let us define A to be the set of all points where Player One chooses the pure strategy L, and B the set where Player One chooses the pure strategy R. By assumption both A and B are measurable sets and by Lemma 5, $T(A) = B$ and $T(B) = A$ (modulo sets of measure zero). By the ergodicity of T^2 we must have that $\mu(A) = 0$ or $\mu(A) = 1$. Since A and B are sets of the same measure, we must conclude that both are of measure zero. We proceed the same way with the sets where Players Two and Three both choose either L or R. By Lemma 5 we have exhausted the abnormal cells. ∎

THEOREM 1: *(f^1, f^2, f^3), an arbitrary Bayesian equilibrium, could not have consisted of measurable functions (and therefore there could not have been a Harsanyi equilibrium by Proposition 2).*

Proof: The result follows directly from Propositions 3 and 4. ∎

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References

[1] P. Billingsley, *Ergodic Theory and Information*, Wiley, New York, 1965.

[2] L. Breiman, *Probability*, Classics in Applied Mathematics, SIAM, Philadelphia, 1992.

[3] R. M. Dudley, *Real Analysis and Probability*, Wadsworth and Brooks/Cole, Pacific Grove, California, 1989.

[4] J. C. Harsanyi, *Games with incomplete information played by Bayesian players: Parts I, II, III*, Management Science **14** (1967–8), 3, 5, 7.

[5] J.-F. Mertens and S. Zamir, *Formulation of Bayesian analysis for games with incomplete information*, International Journal of Game Theory **14** (1985), 1–29.

[6] J.-F. Mertens, S. Sorin and S. Zamir, *Repeated Games*, Core Discussion Papers 9420-22, Université Catholique de Louvain, 1994.

[7] P. Milgrom and R. Weber, *Distribution strategies for games with incomplete information*, Mathematics of Operations Research **10** (1985), 619–632.

[8] J. Nash, *Equilibrium points in n-person games*, Proceedings of the National Academy of Sciences **36** (1950), 48–49.

[9] H. Nikaido, *On von Neumann's minimax theorem*, Pacific Journal of Mathematics **4** (1954), 65–72.

[10] R. Simon, *The Common Prior Assumption in Belief Spaces: An Example*, Discussion Paper 228, Center for Rationality and Interactive Decision Theory, The Hebrew University of Jerusalem, Jerusalem, December 2000.

[11] M. Sion, *On general minimax theorems*, Pacific Journal of Mathematics **8** (1958), 171–176.

[12] R. Solovay, *A model of set-theory in which every set of reals is Lebesgue measurable*, Annals of Mathematics **92** (1970), 1–56.

[13] S. Sorin and S. Zamir, *A two-person game with lack of information on one and one-half sides*, Mathematics of Operations Research **10** (1985), 17–23.

[14] P. Walters, *An Introduction to Ergodic Theory*, Graduate Texts in Mathematics 79, Springer-Verlag, Berlin, 1982.